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Finite Element Analysis of Cracking Bodies

C. K. Gunther*
Boeing Military Airplane Co., Seattle, Wash.
and
K. A. Holsapple† and A. S. Kobayashi‡
University of Washington, Seattle, Wash.

A singular finite element has been developed for the solution of time-dependent stress intensity factors for stationary and moving cracks in finite bodies. The finite element formulation of elastodynamics, including a detail description of the design of the singular element, is presented. The element matrices are derived from a variational principle involving independent interior and boundary displacements and tractions. The interior displacement functions consist of terms of the constant velocity crack solution. The coefficient of the lowest-order term provides the explicit solution of the stress intensity factor. Boundary displacements are selected to provide continuity between singular and regular elements. The discretized equations are solved by the implicit method of temporal integration. Two methods of crack propagation are investigated. The crack tip moves inside the singular element in one method; and, in the other, the crack tip remains fixed in the element and crack propagation is accomplished by moving the entire element. Solutions to static and elastodynamic problems involving stationary and propagating cracks are presented and compared to other widely accepted solutions. The accuracy of the finite element procedure has been found to be good and continued development is in progress.

Introduction

MUCH interest exists in describing the stress and displacement field in the vicinity of crack tips in cracked material bodies. A parameter, the stress intensity factor, is capable of characterizing the near-field displacements and stresses and allows judgements to be made concerning the behavior of such cracks under static and dynamic loads. The solution of stress intensity factors for a variety of load conditions therefore has been the subject of many research efforts. Analytical solutions exist only for simple loads in infinitely large bodies. For problems involving finite boundaries, numerical techniques must be employed to obtain solutions to the governing equations, or to at least extract the stress intensity factor from these equations. A popular method for the solution of problems in static elasticity and in elastodynamics is the finite-element method. This method is particularly suited for bodies with irregular shaped boundaries. A number of techniques are available to obtain the stress intensity factor from such solutions. One method of special interest in the context of this work is the use of singular elements that contain the near-field displacement or stress functions as approximating functions. With these elements it is possible to solve for the stress intensity factor directly, together with the unknown displacements or stresses.

Among the first attempts at solving for dynamic stress intensity factors for propagating cracks by the finite element method were those by King et al.[1] Displacement eigenfunctions associated with a crack propagating at constant velocity were utilized in the procedure. The stress intensity factors were obtained by matching nodal displacements in the vicinity of the crack tip with the displacement eigenfunctions. Reasonably good results were achieved for higher crack tip velocities for the case analyzed earlier by Broberg.[2] Later Anderson et al.[3] used an eight-noded singular element that employed the first 13 terms of William's eigenfunctions for dynamic crack opening problems. Problems of stationary cracks were solved successfully by this approach. Later they used this element to solve problems of propagating cracks.[4] The crack was allowed to propagate inside the element for a specific distance. The element was then shifted in the direction of the crack propagation when the crack tip had reached an extreme forward position. Severe oscillations with increasing amplitudes in the stress intensity factors for the Broberg problem were encountered. The authors did not discuss the type of mass matrix used in connection with the singular element. King and Malluck[5] reported on another attempt at using the same singular element just described. A solution to a problem analyzed by Baker[6] again shows large amplitudes in the oscillations after shifting the singular element.

A more successful use of a singular element was reported by Aoki et al.[7] A seven-noded singular element was used for which the stiffness and mass matrices were derived from the near-field displacement functions of a propagating crack. Also a "damping" matrix was included in the derivation. This results from convection terms of time derivatives of the displacement functions that are expressed with reference to a coordinate system moving with the crack tip. The authors presented the first comprehensive derivation of the finite element solution including singular element to elastodynamic problems. Antiplane shear and crack opening modes of deformations were considered. Although the oscillations in the stress intensity factors common to the earlier attempts are no longer present, the accuracy of the solutions described is not satisfying. In fact, experimentations by these authors with regular elements and a COD-approach for the solution of the stress intensity factor yielded better results for stationary cracks. The poor accuracy of the approach by Aoki et al. must be the result of, first, using only the lowest-order term in the approximating functions and, second, the lack of continuity in the internodal displacements between the singular and adjacent regular elements.

Atluri et al.[8] developed and successfully used a singular element based on the displacement eigenfunctions for the constant velocity crack. In order to provide continuity in the displacements, velocities, and accelerations between the

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*Engineer, Advanced Airplane Branch.

†Associate Professor, Department of Aeronautics and Astronautics.

‡Professor, Department of Mechanical Engineering.

singular and adjacent regular elements, they minimize error integrals involving these quantities. Thus, compatibility is achieved in the least squares sense. The singular element is defined by ten nodes and the crack tip remains fixed at the center of the edge located along the crack surface. Crack propagation is achieved by moving the entire element and by distorting adjacent regular elements by amounts appropriate for the size of the timestep and the crack tip velocity. The finite-element mesh about the singular element is redefined when the distortion of the regular elements reaches a certain degree of severity. A consistent mass matrix and damping matrix are derived and an implicit time integration scheme is employed to obtain solutions to two-dimensional problems of stationary and propagating cracks. The solutions presented agree well with solutions in the literature.

This paper presents a further approach to using a singular finite element for the solution of dynamic stress intensity factors for stationary as well as propagating cracks. The following paragraphs describe the method of solution employed, and present results obtained in the investigation of its feasibility and accuracy.

Formulation of Problem

The governing equations are the equations of linear elastodynamics. A discussion of the derivation of these equations as they apply to cracking bodies is presented by Eftis and Liebowitz,[9] and will not be repeated here. The material is assumed to be homogeneous and isotropic. The problems for which solutions are sought involve a priori assumed direction of crack propagation and crack propagation velocity. The solution is found from the condition that

$$W=\int_{t_0}^{t_1}\left\{\int_R\left[\frac{1}{2}t_{ij}\epsilon_{ij}-\rho b_i u_i-\frac{1}{2}\rho\dot{u}_i\dot{u}_i\right]\mathrm{d}v-\int_{S_t}t_i^0 u_i\,\mathrm{d}s\right\}\mathrm{d}t \quad (1)$$

is stationary for continuous functions u_i (X_1, X_2, X_3, t) satisfying boundary and initial conditions. The strains ϵ_{ij} are calculated from the geometric relations, the velocities $\dot{u}_i$ are obtained from u_i by differentiation, and the stresses t_{ij} are obtained from the strains by using the constitutive relation. The body-force components are denoted by b_i, the mass density by ρ, and prescribed traction components by t_i^0.

The region R of the body is divided into a finite number of subregions R_n, $n=1,2,...,p$ for the purpose of finite-element formulation. The body is in a state of plane strain. The crack tips are embedded in one of the subregions, say, R_1 and R_2. Subregions R_n, $n=3,4,...,p$ may border on part of the crack surface. For convenience, the subregions are assumed to be quadrilaterals and the exterior boundary of R is assumed to consist of straight line segments. Approximate solutions that are discontinuous across subregions will be considered. In this case, the functional Eq. (1) must be generalized into a so-called hybrid functional[10,11] defined by

$$\pi=\int_{t_0}^{t_1}\sum_{n=1}^{p}\left\{\int_{R_n}\left[\frac{1}{2}t_{ij}\epsilon_{ij}-\rho b_i u_i-\frac{1}{2}\rho\dot{u}_i\dot{u}_i\right]\mathrm{d}v\right.$$
$$\left.+\int_{S_{\mathrm{in}}}t_i(v_i-u_i)\,\mathrm{d}s-\int_{S_{tn}}t_i^0 v_i\,\mathrm{d}s\right\}\mathrm{d}t \quad (2)$$

Here S_{in} are interior interfaces between subregions, excluding the crack surface but including S_{un}, the part of the subregion for which displacements are prescribed. Approximating displacement functions u_i are assumed in the interior of each subregion or finite element, approximating displacement functions v_i and tractions t_i are assumed along the boundary of each element. The integral

$$\int_{S_{\mathrm{in}}}t_i(v_i-u_i)\,\mathrm{d}s$$

in Eq. (2) represents a constraint condition on functions u_i and v_i along the element interfaces, whereby the tractions t_i assume the role of Lagrange multipliers. The constraint integral provides for continuity between elements for which different approximating functions u_i will be assumed. Boundary displacements v_i will be assumed as linear functions of the nodal displacements of each element.

The condition that the functional π be stationary for admissible functions $u_i(X,Y,t)$, $v_i(s,t)$ and $t_i(s,t)$, where s is an arc length parameter along the boundary of an element, is equivalent to the momentum equation of linear elastodynamics, the boundary conditions, and the condition that

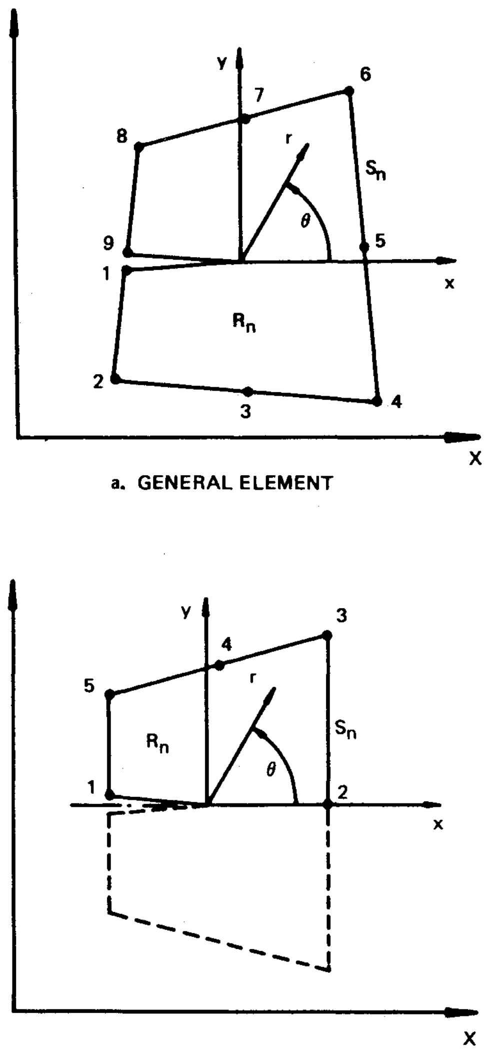
$$t_i=t_{ij}n_j \quad (3)$$

$$u_i=v_i \text{ on } S_n \quad (4)$$

The process of developing the discretized set of equations corresponding to $\delta\pi=0$ is well known and reported in the literature (for example, Ref. 12). Regular elements used in the present method are four-noded isoparametric elements. The selected functions and resulting element matrices for the elements coinciding with the crack tip regions are dealt with in detail in the following section.

Singular Element

A singular finite element has been developed for elastodynamic problems of cracked bodies. This element has been designed to be used in conjunction with four-noded



a. GENERAL ELEMENT

b. SYMMETRIC-HALF ELEMENT

Fig. 1 Singular finite element.

regular elements, but is not restricted to such applications. An extension to higher-order elements is straightforward. Continuity is preserved along common boundaries between singular and regular elements by the assumption of independent boundary functions. The inner-element displacement functions include the near-field terms of order $r^{1/2}$. Coefficients of these terms provide a direct solution of the stress intensity factor. The crack tip is free to propagate inside the element. Nine nodes are used to define the general singular element (Fig. 1a). It has a symmetric-half version described by five nodes (Fig. 1b). The approximating functions for this element are written in the forms

$$u = U_1\beta_1 + U_2\beta_2 \tag{5}$$

$$v = Lq \tag{6}$$

$$t = R_1\alpha_1 + R_2\alpha_2 \tag{7}$$

where α_1, α_2, β_1, β_2 are unknown coefficients; the components of U_1, R_1 are polynomials satisfying the dynamic linear-momentum equation. These polynomials are derived from the solution to the constant-crack-velocity problem and are described in detail in Ref. 14. Terms U_2 and R_2 correspond to rigid body motion. Functions L are so chosen that the displacements v match displacements of adjacent regular elements. The components of u, v, and t are u_i, v_i, and t_i, respectively. The velocities are derived as

$$\dot{u} = \dot{U}_1\beta_1 + U_1\dot{\beta}_1 + U_2\dot{\beta}_2 \tag{8}$$

$$\dot{v} = L\dot{q} \tag{9}$$

the strains are

$$\epsilon = E\beta_1 \tag{10}$$

and the stresses

$$\tau = D\epsilon \tag{11}$$

The components of ϵ and τ are ϵ_{ij} and t_{ij}, respectively. Matrix E contains derivatives of U_1 and D is the elasticity matrix for plane strain. Writing Eq. (2) in the form

$$\pi = \sum_{n=1}^{p} \pi_n \tag{12}$$

where

$$\pi_n = \int_{t_0}^{t_1} \left\{ \int_{R_n} \left[\frac{1}{2} t_{ij}\epsilon_{ij} - \rho b_i u_i - \frac{1}{2}\rho \dot{u}_i \dot{u}_i \right] dv + \int_{S_{in}} t_i (v_i - u_i)\, ds - \int_{S_{tn}} t_i^0 v_i\, ds \right\} dt \tag{13}$$

and substituting Eqs (5-8), (10), and (11) into Eq. (13) yields the expression

$$\pi_n = \int_{t_0}^{t_1} \left[\frac{1}{2}\beta_1^T H\beta_1 - \beta_1^T F_1 - \beta_2^T F_2 - \frac{1}{2}\beta_1^T M_1 \beta_1 - \frac{1}{2}\dot{\beta}_1^T M_2 \dot{\beta}_1 - \frac{1}{2}\dot{\beta}_2^T M_3 \dot{\beta}_2 - \beta_1^T M_4 \dot{\beta}_1 - \dot{\beta}_1^T M_5 \dot{\beta}_2 - \beta_1^T M_6 \dot{\beta}_2 + \alpha_1^T G_1 q + \alpha_2^T G_2 q - \alpha_1^T P_1 \beta_1 - \alpha_2^T P_2 \beta_2 - \alpha_2^T P_3 \beta_1 - \alpha_1^T P_4 \beta_2 - qS \right] dt \tag{14a}$$

The following integrals were introduced for convenience:

$$H = \int_{R_n} E^T D E\, dv$$

$$F_1 = \int_{R_n} \rho U_1^T b\, dv \qquad F_2 = \int_{R_n} \rho U_2^T b\, dv$$

$$M_1 = \int_{R_n} \dot{U}_1^T \rho \dot{U}_1\, dv \qquad M_2 = \int_{R_n} U_1^T \rho U_1\, dv$$

$$M_3 = \int_{R_n} U_2^T \rho U_2\, dv \qquad M_4 = \int_{R_n} \dot{U}_1^T \rho U_1\, dv$$

$$M_5 = \int_{R_n} U_1^T \rho U_2\, dv \qquad M_6 = \int_{R_n} \dot{U}_1^T \rho U_2\, dv$$

$$M_7 = \int_{R_n} \ddot{U}_1^T \rho U_1\, dv \qquad M_8 = \int_{R_n} \ddot{U}_1^T \rho U_2\, dv$$

$$G_1 = \int_{S_{in}} R_1^T L\, ds \qquad G_2 = \int_{S_{in}} R_2^T L\, ds$$

$$P_1 = \int_{S_{in}} R_1^T U_1\, ds \qquad P_2 = \int_{S_{in}} R_2^T U_2\, ds$$

$$P_3 = \int_{S_{in}} R_2^T U_1\, ds \qquad P_4 = \int_{S_{in}} R_1^T U_2\, ds$$

$$S = \int_{S_{tn}} L^T t^0\, ds \tag{14b}$$

An error in Ref. 15 has been corrected here by including the term $\alpha_1^T P_4 \beta_2$.

The coefficients α_1, α_2, β_1, β_2 can be varied independently for each element and the condition $\delta\pi_n = 0$ yields

$$H\beta_1 - F_1 - M_1\beta_1 + \dot{M}_2\dot{\beta}_1 + M_2\ddot{\beta}_1 - M_4\dot{\beta}_1 + \dot{M}_4^T\beta_1 + M_4^T\dot{\beta}_1 + \dot{M}_5\dot{\beta}_2 + M_5\ddot{\beta}_2 - M_6\dot{\beta}_2 - P_1^T\alpha_1 - P_3^T\alpha_2 = 0 \tag{15}$$

$$-F_2 + \dot{M}_3\dot{\beta}_2 + M_3\ddot{\beta}_2 + \dot{M}_5^T\dot{\beta}_1 + M_5^T\ddot{\beta}_1 + \dot{M}_6^T\beta_1 + M_6^T\dot{\beta}_1 - P_2^T\alpha_2 - P_4^T\alpha_1 = 0 \tag{16}$$

$$G_1 q - P_1\beta_1 - P_4\beta_2 = 0 \tag{17}$$

$$G_2 q - P_2\beta_2 - P_3\beta_1 = 0 \tag{18}$$

Equations (15-18) can be used to solve for α_1, α_2, β_1, β_2 in terms of q. Assuming that the number of coefficients α_1 equals the number of β_1, the number of α_2 equals the number of β_2, and that P_1 is invertible, one obtains from Eqs. (17) and (18)

$$\beta_1 = Bq \tag{19}$$

$$\beta_2 = Aq \tag{20}$$

$$A = (P_3 P_1^{-1} P_4 - P_2)^{-1} (P_3 P_1^{-1} G_1 - G_2) \tag{21}$$

$$B = P_1^{-1} (G_1 - P_4 A) \tag{22}$$

It is not necessary to solve for α_1 and α_2 explicitly. An examination of Eqs. (14) shows that all terms involving α_1 and α_2 vanish with β_1 and β_2 expressed by Eqs. (19) and (20), respectively.

Substituting Eqs. (19) and (20) into Eqs. (14) yields the following expression for the functional π_n:

$$\pi_n = \int_{t_0}^{t_1} \left[\tfrac{1}{2} q^T k_1 q - \tfrac{1}{2} \dot{q}^T m_1 \dot{q} - \dot{q}^T v_1 q - q^T f_s \right] dt \tag{23a}$$

where

$$k_I = B^T HB - B^T M_I B - \dot{B}^T M_2 \dot{B} - \dot{A}^T M_3 \dot{A} - 2B^T M_4 \dot{B} - 2\dot{B}^T M_5 \dot{A} - 2B^T M_6 \dot{A}$$

$$m_I = B^T M_2 B + A^T M_3 A + 2B^T M_5 A$$

$$v_I = B^T M_2 \dot{B} + A^T M_3 \dot{A} + B^T M_4 B + B^T M_5 \dot{A} + A^T M_5^T \dot{B} + A^T M_6^T B$$

$$f_s = B^T F_I + A^T F_2 + S \tag{23b}$$

Earlier the functional π was introduced in the form Eq. (12) where π_n, for $n=1, 2$, is given by Eqs. (23) and

$$\pi_n = \int_{t_0}^{t_I} \left[\tfrac{1}{2} q^T kq - \tfrac{1}{2}\dot{q}^T m\dot{q} - q^T f \right] \mathrm{d}t \tag{24}$$

for $n = 3,4,\ldots,p$. Matrices k, m, and f are the regular element stiffness, mass, and force matrix, respectively.[12]

Introducing now global nodal displacements q^* and writing the element matrices in corresponding global form yields

$$\pi = \int_{t_0}^{t_I} \left\{ \sum_{n=1}^{2} (\tfrac{1}{2} q^{*T} \bar{K}_I q^* - \tfrac{1}{2}\dot{q}^{*T}\bar{M}_I \dot{q}^* - \dot{q}^{*T}\bar{V}_I q^* - q^{*T} F_s) + \sum_{n=3}^{p} (\tfrac{1}{2} q^{*T}\bar{K}_r q^* - \tfrac{1}{2}\dot{q}^{*T}\bar{M}_r \dot{q}^* - q^{*T} F_r) \right\} \mathrm{d}t \tag{25}$$

The condition that Eq. (25) be stationary with respect to q^* and $\dot{q}^*$ yields

$$\bar{M}\ddot{q}^* + \bar{V}\dot{q}^* + \bar{K}q^* = Q \tag{26a}$$

where

$$\bar{K} = \sum_{n=1}^{2} \bar{K}_s + \sum_{n=3}^{p} \bar{K}_r \qquad \bar{M} = \sum_{n=1}^{2} \bar{M}_s + \sum_{n=3}^{p} \bar{M}_r$$

$$\bar{V} = \sum_{n=1}^{2} \bar{V}_s \qquad Q = \sum_{n=1}^{2} F_s + \sum_{n=3}^{p} F_r \tag{26b}$$

and

$$\bar{K}_s = \tfrac{1}{2}(\bar{K}_I + \bar{K}_I^T) + \dot{\bar{V}}_I \qquad \bar{M}_s = \tfrac{1}{2}(\bar{M}_I + \bar{M}_I^T)$$

$$\bar{V}_s = \tfrac{1}{2}(\dot{\bar{M}}_I + \dot{\bar{M}}_I^T) + \bar{V}_I - \bar{V}_I^T \tag{26c}$$

The singular element matrices corresponding to $\bar{M}_s$, $\bar{V}_s$, $\bar{K}_s$, and F_s can be derived from Eqs. (23b) and (26c). The element matrices are

$$k_s = B^T HB + B^T(M_2\ddot{B} + 2M_4^T\dot{B} + M_5\ddot{A} + M_7^T B) + A^T(M_3\ddot{A} + 2M_6^T\dot{B} + M_5^T\ddot{B} + M_8^T B) \tag{27}$$

$$m_s = B^T M_2 B + A^T M_3 A + B^T M_5 A + A^T M_5 B \tag{28}$$

$$v_s = 2B^T(M_2\dot{B} + M_4^T B + M_5\dot{A}) + 2A^T(M_3\dot{A} + M_6^T B + M_5^T\dot{B}) \tag{29}$$

$$f_s = B^T F_I + A^T F_2 + S \tag{30}$$

Element matrices Eqs. (27-30) can now be calculated by inserting approximating functions into Eqs. (14b) and by performing the integrations. Derivatives of the polynomials in U_I and the polynomials in R_I contain singularities at the crack tip. It is possible therefore that singularities may exist in the integrands of integrals used in Eqs. (27-30). An examination reveals that H, M_6, M_7, and M_8 contain singularities of various orders. Integrating by parts and using the fact that the polynomials in U_I satisfy the linear-momentum equation in the absence of body forces one obtains

$$H + M_7^T = H_I \tag{31}$$

where

$$H_I = \int_{S_n} U_I^T n^T DE \mathrm{d}s \tag{32}$$

Matrices M_6 and M_8 can also be integrated by parts using that

$$\dot{U}_I = -c\frac{\partial}{\partial x} U_I \qquad \ddot{U}_I = c^2 \frac{\partial^2}{\partial x^2} U_I \tag{33}$$

since

$$x = X - ct \qquad y = Y \tag{34}$$

with c, the constant crack tip velocity, yielding

$$M_6 = -c\int_{S_n} U_I^T \rho U_2 \mathrm{d}y \tag{35}$$

$$M_8 = -c\int_{S_n} \dot{U}_I^T \rho U_2 \mathrm{d}y \tag{36}$$

Making use of Eqs. (31), (35), and (36) the element stiffness matrix Eq. (27) changes to

$$k_s = B^T H_I B + B^T(M_2\ddot{B} + 2M_4^T\dot{B} + M_5\ddot{A}) + A^T(M_3\ddot{A} + 2M_6^T\dot{B} + M_5^T\ddot{B} + M_8^T B) \tag{37}$$

and because of absence of body forces Eq. (30) changes to

$$f_s = S \tag{38}$$

Thus all singularities at the crack tip have been eliminated.

The present singular element contains 17 terms in the displacements and tractions. The lowest order polynomial in U_I is of order $r^{1/2}$, therefore the corresponding coefficient β_{II} is related to the stress intensity factor, K_I, by

$$K_I(t) = 3\mu\sqrt{2\pi}\frac{[4s_I s_2 - (1+s_2^2)^2]}{4(1+s_2^2)}\beta_{II}(t) \tag{39}$$

where s_I and s_2 are nondimensional velocity parameters depending on crack tip velocity, and longitudinal and transverse wave velocities. Once the nodal displacements are found, the stress intensity factor can be calculated using Eq. (39).

Solution Procedure

In order to obtain the solution to an elastodynamic problem, one has to solve the discretized set of Eqs. (26). This is accomplished by the implicit method of temporal integration that is developed from finite-difference formulas.[13] The global matrices in Eq. (20) are obtained by summing all element matrices, a process customarily referred to as merging of the element matrices. It can be seen that Eqs. (28) and (37) are not symmetric. In order to be able to use a subroutine for the solution of a set of algebraic equations with symmetric coefficient matrix, matrices $\bar{V}$ and $\bar{K}$ are divided into symmetric and asymmetric parts, i.e.,

$$\bar{V}_{sym} = ½(\bar{V} + \bar{V}^T) \qquad \bar{V}_{asym} = ½(\bar{V} - \bar{V}^T) \qquad (40)$$

with similar formulas for $\bar{K}$. Inserting these expressions into Eq. (26) and using the Newmark - β formulas yields (dropping * from q^* for convenience)

$$K_{eff} q(t+\Delta t) = Q_{eff}(t+\Delta t) \qquad (41a)$$

where

$$\bar{K}_{eff} = \bar{M} + \bar{V}_{sym}(\Delta t/2) + \beta \Delta t^2 \bar{K}_{sym}$$

$$Q_{eff} = \bar{M}[q(t) + \Delta t \dot{q}(t) + (½ - \beta)\Delta t^2 \ddot{q}(t)]$$

$$+ \bar{V}_{sym}[(\Delta t/2) q(t) + (½ - \beta)\Delta t^2 \dot{q}(t) - (¼ - \beta)\Delta t^3 \ddot{q}(t)]$$

$$+ \beta \Delta t^2 [Q(t+\Delta t) - \bar{V}_{asym} \dot{q}(t+\Delta t) - \bar{K}_{asym} q(t+\Delta t)] \qquad (41b)$$

Unfortunately, the correction to Q_{eff} for asymmetry of $\bar{V}$ and $\bar{K}$ requires knowledge of the displacements and velocities about to be solved. Therefore, an iterative procedure is suggested by the form of Eq. (41b). Experimentation by the authors indicated that it is adequate to use the current displacements and velocities. To assure that inaccuracies do not grow with time, infrequent corrections to the solutions are made by calculating an error vector, using Eq. (26), as

$$Q_{err} = Q - \bar{M}\ddot{q} - \bar{V}\dot{q} - \bar{K}q \qquad (42)$$

and subtracting Q_{err} from the external force vector Q in Eq. (41b) and by re-solving the set of equations for an improved solution.

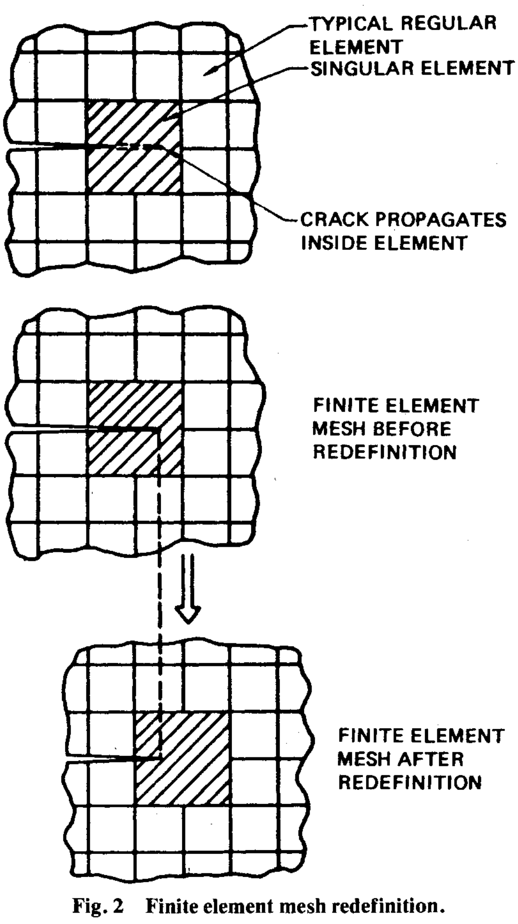


Fig. 2 Finite element mesh redefinition.

The procedure for propagating a crack can be described as follows. The singular element is placed in the finite-element mesh such that the initial crack tip position is within the singular element. The crack tip is then advanced at each timestep by an amount commensurate with the crack tip velocity, prescribed by a crack-tip-velocity history, and the time increment. When the crack tip reaches an extreme forward position, the finite element mesh is redefined locally by advancing the singular element by the distance equal to the width of a regular element (Fig. 2). Therefore, elements that were regular at the previous timestep turn into a singular element and the region of the previous singular element position is filled with regular elements. This procedure, as it is designed presently, requires a certain degree of regularity in the finite-element mesh. In fact, the finite element program is restricted to the solution of problems that are symmetric about the line of crack extension and symmetric about an axis perpendicular to the line of crack extension. These restrictions are not the result of a limitation in the developed procedure, but they are imposed for economic reasons.

Results

The accuracy of the singular element stiffness matrix was first evaluated for zero crack tip velocity by applying near-

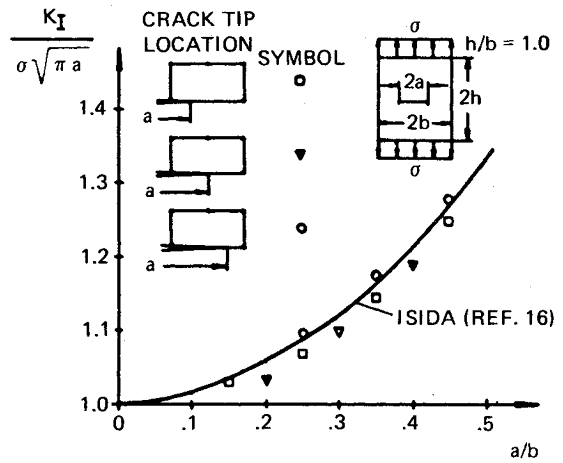


Fig. 3 Stress intensity factor for center-cracked tension plate.

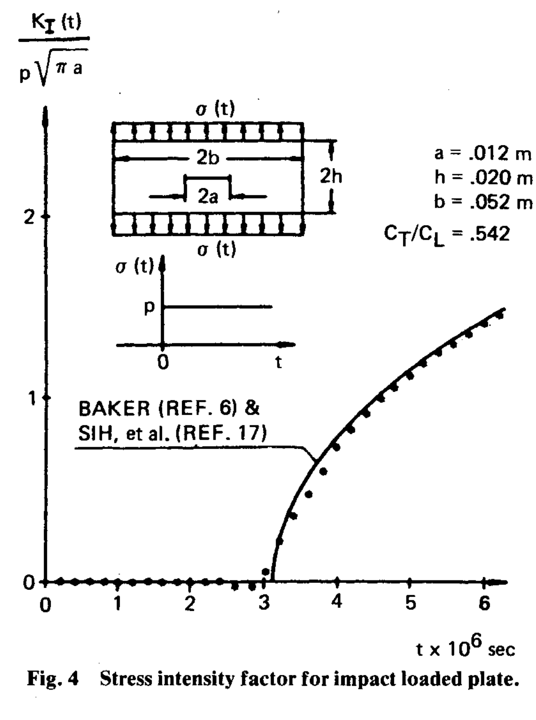


Fig. 4 Stress intensity factor for impact loaded plate.

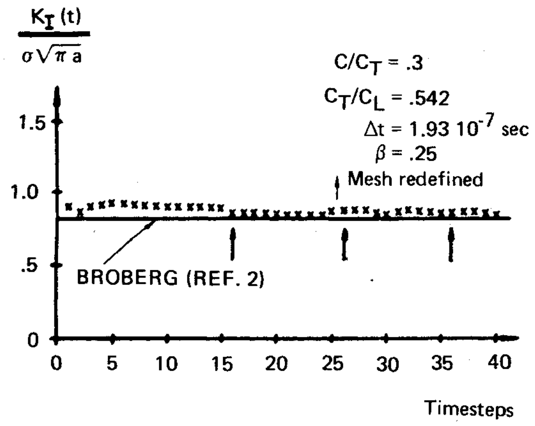


Fig. 5 Stress intensity factor for propagating crack, $c/c_T = 0.3$.

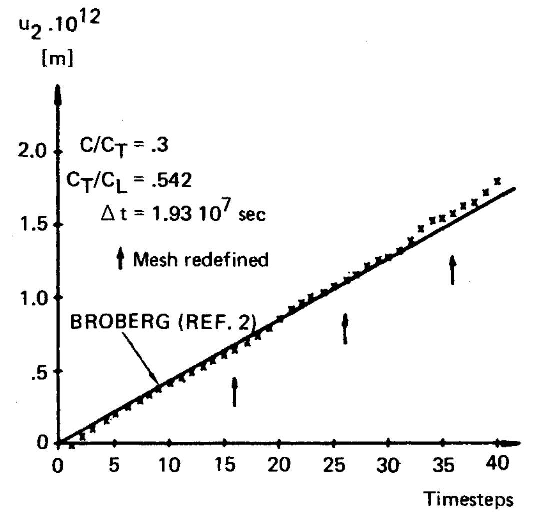


Fig. 6 Displacement at center of crack, $c/c_T = 0.3$.

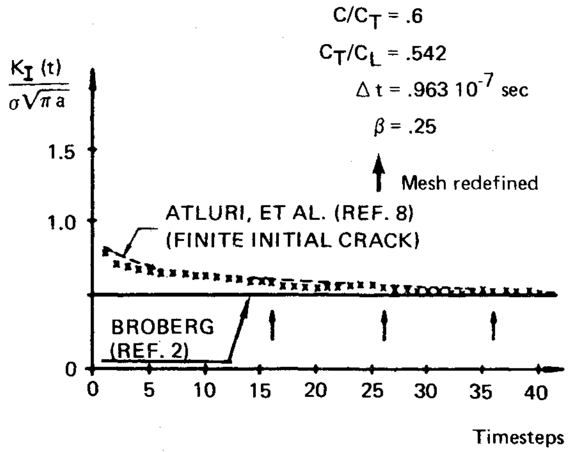


Fig. 7 Stress intensity factor for propagating crack, $c/c_T = 0.6$.

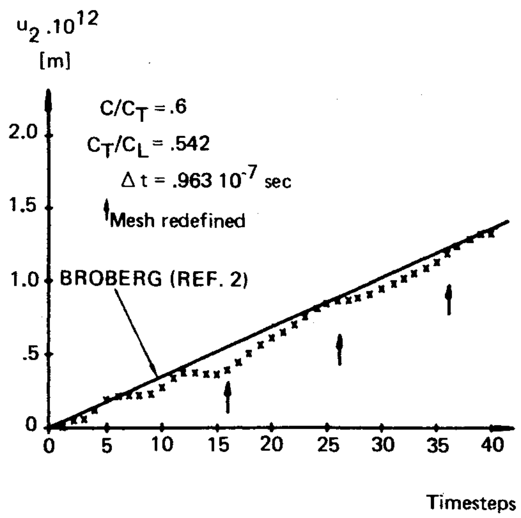


Fig. 8 Displacement at center of crack, $c/c_T = 0.6$.

field nodal forces to the element and recovering the corresponding stress intensity factor. Various crack tip positions within the element were assumed and the stress intensity factor was obtained with errors of less than 0.2% for extreme crack tip positions. The stiffness matrix was further evaluated by solving the static problem of the center-cracked tension plate. Simultaneously, the procedure of redefining the finite-element mesh as the crack is extended was tested. The good agreement achieved with the results obtained by Isida[16] through a boundary collocation procedure is presented in Fig. 3. A characteristic of the crack propagation procedure appears to be the small jump in stress intensity factor for the same crack length but opposite crack tip position within the element.

The accuracy of the derivation of the singular element mass matrix was evaluated by calculating the nodal forces required to suddenly accelerate a rigid body as represented by the element by unit amounts in either direction in the plane of the body. It can be easily shown that the magnitude of the sum of the nodal forces must equal the magnitude of the mass of the body. The mass of the singular element calculated in this fashion was typically obtained with an error of 1.2% in the X and Y directions.

As a first test of the accuracy of the procedure for solving elastodynamic problems, the problem of a stationary crack in a finite size plate subjected to impact along the edges parallel to the crack was solved. This problem was solved by Baker[6] and by Sih et al.[17] Transform methods, including the Wiener-Hopf and Cagniard techniques, were employed to obtain the solutions. The finite-element solution is in excellent agreement with this solution, as shown in Fig. 4.

Next, the more complex elastodynamic problem of propagating cracks was investigated. Broberg[2] solved the problem of a suddenly appearing crack. The crack propagates at constant velocity in a uniform tension field. At first, the case of a crack propagating at 30% of the shear wave velocity was solved. Although Broberg's solution is for an infinite medium, the results apply to a finite-size plate for times until waves reflect from the edges of the plate. The finite element solution starts with the required initial condition of zero crack length. It cannot be expected, however, that accurate stress intensity factors can be obtained for such an extreme crack tip position within the singular element; therefore, a larger error is expected initially. The finite element solution (Fig. 5) exhibits such an error initially, but then approaches the solution given by Broberg. The finite-difference equations were solved with $\beta = 1/4$. The time increment Δt was selected such that the crack moved 0.2 mm per timestep. The solution was iterated once each time the mesh was redefined as indicated by arrows in Fig. 5. Figure 6 shows the crack opening displacement at the center of the crack obtained from the finite element solution being also in excellent agreement with Broberg's result.

Next, the case of a crack propagating at 60% of the shear wave velocity was solved. Again, the stress intensity factor initially shows a larger error but then converges to the correct solution (Fig. 7). The results also agree well with those of Ref. 8. The crack opening displacement at the center of the crack is shown in Fig. 8. It must be mentioned here that the results shown here are improved over those in Ref. 15. This is due to the corrected error in Eq. (14) as mentioned earlier.

Overall, the agreement with Broberg's results is good. The finite element results exhibit a small degree of oscillation in the stress intensity factors and in the displacements. Atluri et al.[8] used a different crack propagation method and did not show such oscillations in their results. Their procedure is based on a fixed crack tip position within the singular element and the crack is propagated by moving the singular element and by deforming adjacent regular elements at each timestep. Interpolations are required at each timestep since the singular element nodes change position. This method was briefly investigated.[14] The resulting stress intensity factors agree well with Ref. 8 and exhibit the same degree of smoothness.

Discussion and Conclusion

The singular finite element can be used with success to the accurate solution of time dependent stress intensity factors for stationary and propagating cracks. The results indicate that the fixed-crack-tip method for crack propagation yields the better results of the two methods. Further experimentation is, however, justified with the variable-crack-tip method since it offers the capability of modeling shorter initial crack lengths. Use of higher order elements or increasing the size of the singular element relative to the regular element mesh size are considered for further development. The applicability of the singular element to variable crack velocity problems is under investigation.

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